Explicit Conjunctions w/o Compilation: Computing $h^{\textbf{FF}}(\Pi^C)$ in Polynomial Time (Technical Report)

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Abstract

A successful partial delete relaxation method is to compute h^{FF} in a compiled planning task Π^C which represents a set C of conjunctions explicitly. While this compilation view of such partial delete relaxation is simple and elegant, its meaning with respect to the original planning task is opaque. We provide a direct characterization of $h^+(\Pi^C)$, without compilation, making explicit how it arises from a "marriage" of the critical-path heuristic h^m with (a somewhat novel view of) h^+. This explicit view allows us to derive a direct characterization of $h^{\text{FF}}(\Pi^C)$, which in turn allows us to compute a version of that heuristic function in time polynomial in $|C|$.

Introduction

Explicit conjunctions were first introduced (Haslum 2009) to characterize critical-path heuristics (Haslum and Geffner 2000) as $h^m = h^1(\Pi^m)$, where Π^m is a compiled task representing each m-conjunction c via a newly introduced *π-fluent* π_c. A modified compilation Π^C (Haslum 2012), for arbitrary sets C of conjunctions, was shown to yield a *partial delete relaxation* method, guaranteeing to *converge* to h^*, i. e., $h^+(\Pi^C) = h^*$ for appropriately chosen C. The size of Π^C is worst-case exponential in $|C|$ because it explicitly enumerates every subset $C' \subseteq C$ of conjunctions that any application of an action a from the original planning task may be used to support. This size explosion was tackled by the Π^C_{ce} compilation (Keyder, Hoffmann, and Haslum 2012; 2014), which handles each c by a separate conditional effect. Π^C_{ce} still guarantees convergence, but loses information as it ignores *cross-context* conditions, i. e. precondition π-fluents which arise only from the combination of several $c \in C'$.

We provide a direct formulation, without compilation, of *delete relaxation over explicit conjunctions*. This makes explicit some previously opaque aspects of the approach, in particular explaining the complexity difference between Π^C and Π^C_{ce} in terms of a subgoal-choice problem easy for Π^C_{ce} but hard for Π^C. By solving that problem greedily, we compute relaxed plans for Π^C in time polynomial in $|C|$. This supersedes Π^C_{ce}, in terms of achieving the same complexity reduction without having to ignore cross-context conditions. (Alcazar et al. (2013) pursued a similar direction but, as we will detail, trivialized the subgoaling and lost convergence.)

Most proofs are moved out of the main text. They are available in an appendix.

Preliminaries

We employ the usual STRIPS syntax and semantics. Planning tasks are tuples $\Pi = (F, A, I, G)$ of facts, actions, initial state, and goal, each $a \in A$ being a triple $(pre(a), add(a), del(a))$ where $add(a) \cap del(a) = \emptyset$. For simplicity, we consider uniform costs (all action costs are 1); our results straightforwardly extend to arbitrary action costs.

We assume the standard notions of h^+ and h^*. We characterize heuristic functions in terms of equations over regressed subgoals. The *regression of fact set g over action a*, $R(g, a)$, is defined if $add(a) \cap g \neq \emptyset$ and $del(a) \cap g = \emptyset$. In that case, $R(g, a) = (g \setminus add(a)) \cup pre(a)$; otherwise, we write $R(g, a) = \bot$. To simplify notation, we will often identify a heuristic h with its value $h(I)$ in the initial state. All statements made generalize to arbitrary states s by setting $I := s$. By $h(\Pi')$, we denote a heuristic for Π whose value is given by applying h in a modified task Π'. To make explicit that h is computed on Π itself, we write $h(\Pi)$.

Let C be a set of fact conjunctions. We identify conjunctions with fact sets. We assume throughout that C contains all singleton conjunctions, $\{p\}$ for $p \in F$. The Π^C compilation and its relatives are based on introducing a *π-fluent* π_c for each $c \in C$. Using the shorthand $X^C := \{\pi_c \mid c \in C \wedge c \subseteq X\}$ for fact sets X, Π^C can be defined as follows:

Definition 1 *Π^C is the planning task (F^C, A^C, I^C, G^C), where A^C contains an action $a^{C'}$, for every pair $a \in A$ and $\emptyset \neq C' \subseteq \{c \in C \mid R(c, a) \neq \bot\}$, with $a^{C'}$ given by $pre(a^{C'}) = (pre(a) \cup \bigcup_{c' \in C'}(c' \setminus add(a)))^C$, and $add(a^{C'}) = \{\pi_{c'} \mid c' \in C'\}$. Π^C_{nc} is identical to Π^C except that $pre(a^{C'}) = pre(a)^C \cup \bigcup_{c' \in C'}(pre(a) \cup (c' \setminus add(a)))^C$.*

This changes Haslum's (2012) definition in minor ways, simplifying our presentation without affecting our results. $\emptyset \neq C'$ and $add(a^{C'}) = \{\pi_{c'} \mid c' \in C'\}$ work as C contains all singleton conjunctions. A *cross-context* condition for $a^{C'}$ is a $c \in C$ where $c \subseteq pre(a) \cup \bigcup_{c' \in C'}(c' \setminus add(a))$ but there exists no single $c' \in C'$ s.t. $c \subseteq pre(a) \cup (c' \setminus add(a))$. Π^C_{nc} ignores cross-context conditions, and is equivalent to the Π^C_{ce} compilation in that $h^+(\Pi^C_{ce}) = h^+(\Pi^C_{nc})$.

$h^+(\Pi^C)$ w/o Compilation

We commence our investigation by "marrying" critical-path heuristics with the optimal delete-relaxation heuristic h^+. This results in a new characterization of $h^+(\Pi^C)$. Critical-path heuristics here will be captured by a straightforward extension h^C of the standard heuristic h^m to consider an arbitrary conjunction set C. We capture h^+ in terms of a novel equation suitable for the subsequent "marriage".

We define $h^C := h(G)$ where $h(.)$ is the function on fact sets g that satisfies $h(g) =$

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, R(g,a) \neq \bot} h(R(g,a)) & g \in C \\ \max_{g' \subseteq g, g' \in C} h^C(g') & \text{else} \end{cases} \quad (1)$$

It is easy to see that h^C extends h^m, in the sense that $h^C = h^m$ if C consists of all conjunctions of size $\leq m$.

We characterize h^+ as $h^+ := h(G)$ where $h(.)$ is the function on fact sets g that satisfies $h(g) =$

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' = \{p \in g \mid R(\{p\},a) \neq \bot\}} \\ \quad h((g \setminus g') \cup \bigcup_{p \in g'} R(\{p\}, a)) & \text{else} \end{cases} \quad (2)$$

We prove this correct further below. To understand it intuitively, note that, for singleton subgoals, regression ignores the delete list: $R(\{p\}, a) \neq \bot$ iff $p \in add(a)$. Hence, in the bottom case, we will always have $g' = add(a) \cap g \neq \emptyset$. As, furthermore, $R(\{p\}, a) = pre(a)$, the recursive call of h will always be on $(g \setminus add(a)) \cup pre(a)$.

While the notation in Equation 2 is cumbersome, it (links to the general case below and) makes visible that the delete relaxation can be understood as *splitting subgoals up into singletons, and considering regression separately with respect to each of these*. Because regression over singleton facts ignores the deletes as outlined, in effect we need to worry only about the part of the subgoal we can support, not about other parts that the same action may contradict. This understanding underlies our "marriage of h^+ with h^C": instead of the singleton facts in our subgoal, we now have to achieve (apply regression to) its C-conjunctions.

Definition 2 *The* explicit-C delete relaxation heuristic h^{C+} *is defined as* $h^{C+} := h(\{G\})$*, where* $h(.)$ *is the function on conjunction sets* $\mathcal{G}$ *that satisfies* $h(\mathcal{G}) =$

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}} \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g,a)\}) & \forall g \in \mathcal{G} : g \in C \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, g' \in C\}) & \textit{else} \end{cases}$$

h^{C+}_{nc} *is identical to* h^{C+} *except that, in the middle case,* $\{R(g,a) \mid g \in \mathcal{G}'\}$ *is used instead of* $\{\bigcup_{g \in \mathcal{G}'} R(g,a)\}$.

Note how Definition 2 merges the ideas underlying h^C and h^+: We have to support *atomic subgoals* from C individually by regression (h^C); but instead of achieving only the most costly one, we have to achieve all of them, though separately (h^+). Where, previously, atomic subgoals were just facts (and hence we dealt with a set g of facts in the recursion), they now are conjunctions (hence the set $\mathcal{G}$ of conjunctions in the recursion). Akin to Equation 1, the bottom case serves to extract the atomic subgoals (defined by our conjunction set C) from a non-atomic subgoal. Akin to Equation 2, the middle case selects the best action supporting some of our atomic subgoals. Note that using action a to "support" a subgoal g, which is no longer necessarily a singleton, means to comply with the full definition of regression: a is not allowed to delete any fact in g, and $R(g, a) = (g \setminus add(a)) \cup pre(a) \supseteq pre(a)$.

Importantly, in difference to Equation 2, there is now a choice of $\mathcal{G}'$. This is the "subgoal-choice problem" mentioned in the introduction. The pairs $a, \mathcal{G}'$ here correspond to the pairs a and C' in the actions $a^{C'}$ of Π^C and Π^C_{nc}. The choice is needed because (a) for Π^C (but not for Π^C_{nc}), larger $\mathcal{G}'$ may give rise to additional cross-context conditions; and because (b) for both Π^C and Π^C_{nc}, it may be advantageous to achieve an atomic subgoal g later on in the recursion, with an action that has a different precondition which is easier to combine with g. To illustrate (a), say that $add(a) = \{p\}$ and $\mathcal{G} = \{\{p, q_1\}, \{p, q_2\}\}$ where q_1 and q_2 are impossible to achieve together. Then $\mathcal{G}' = \{\{p, q_1\}, \{p, q_2\}\}$ in h^{C+} (corresponding to Π^C) leads to the unsolvable subgoal $\{\{q_1, q_2\}\}$, while $\mathcal{G}' = \{\{p, q_1\}\}$ leads to the subgoal $\{\{q_1\}, \{p, q_2\}\}$ which is solvable because we can achieve each of $\{q_1\}$ and $\{p, q_2\}$ separately. To illustrate (b), say again that $\mathcal{G} = \{\{p, q_1\}, \{p, q_2\}\}$. Say that $add(a) = \{p\}$ and $pre(a) = \{r\}$ where the conjunction $\{r, q_1\}$ takes a single step to achieve, but $\{r, q_2\}$ takes $N > 2$ actions to achieve. Say that $add(a') = \{p\}$ and $pre(a) = \{r'\}$ where the conjunction $\{r', q_2\}$ takes a single step to achieve, but $\{r', q_1\}$ takes $N > 2$ actions to achieve. Then using $\mathcal{G}' = \{\{p, q_1\}, \{p, q_2\}\}$ for either of a or a' yields a solution of length $N + 2$, while using $\mathcal{G}' = \{\{p, q_1\}\}$ for a and, subsequently, $\mathcal{G}' = \{\{p, q_2\}\}$ for a', yields a solution of length 4. So the only optimal solutions are such that neither a nor a' make maximal use of the subgoals they could support.

To prove that Definition 2 does indeed capture $h^+(\Pi^C)$, i. e., that $h^+(\Pi^C) = h^{C+}(\Pi)$, we start with the simple case where C contains only the singleton conjunctions:

Lemma 1 *For* $C = \{\{p\} \mid p \in F\}$*,* $h^+ = h^{C+}$*.*

Proof Sketch: For $C = \{\{p\} \mid p \in F\}$, $\mathcal{G}$ will always be a set of singleton conjunctions, which we can identify with a set g of facts. Re-writing the h^{C+} equation yields:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' \subseteq \{p \in g \mid R(\{p\},a) \neq \bot\}} \\ \quad h((g \setminus g') \cup \bigcup_{p \in g'} R(\{p\}, a)) & \text{else} \end{cases}$$

Choosing $g' \subset \{p \in g \mid R(\{p\}, a) \neq \bot\}$ can only lead to a larger subgoal $(g \setminus g') \cup pre(a)$, hence we can exclude these choices and the equation simplifies to Equation 2. Simplifying the notations, that becomes:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' = g \cap add(a)} \\ \quad h((g \setminus add(a)) \cup pre(a)) & \text{else} \end{cases}$$

This last equation corresponds to h^+, proving the claim. ■

Theorem 1 $h^+(\Pi^C) = h^{C+}(\Pi)$.

Proof Sketch: Denoting h^{C+} for singleton conjunctions only by h^{1+}, with Lemma 1 the claim is equivalent to $h^{1+}(\Pi^C) = h^{C+}(\Pi)$. We prove this by comparing two equations, capturing $h^{1+}(\Pi^C)$ respectively $h^{C+}(\Pi)$.

For $h^{1+}(\Pi^C)$, after some simplifications our equation (called *Equation I*) reads as follows:

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I^C \\ 1 + \min_{a^{\mathcal{G}'} \in A^C, \emptyset \neq \mathcal{G}' \subseteq \{\{\pi_g\} \in \mathcal{G} \mid R(g,a) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup pre(a^{\mathcal{G}'})) & \forall g \in \mathcal{G} : |g| = 1 \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, |g'| = 1\}) & \text{else} \end{cases}$$

For $h^{C+}(\Pi)$, we equivalently modify Definition 2 into an equation (called *Equation II*) that works on *completed* subgoals $\mathcal{G}$ only. $\mathcal{G}$ is *completed* if, for all $g \in \mathcal{G}$, every $g' \subseteq g$ with $g' \in C$ is contained in $\mathcal{G}$ as well.

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g,a)\}) & \\ \qquad \mathcal{G} \text{ is completed and } \forall g \in \mathcal{G} : g \in C & \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, g' \in C\}) & \text{else} \end{cases}$$

Equations I and II are isomorphic because Equation II works on C-subgoals and Equation I works on singleton π-fluents representing these same C-subgoals. Spelling this out is (notationally cumbersome but) straightforward. ∎

A similar proof shows that $h^+(\Pi^C_{nc}) = h^{C+}_{nc}(\Pi)$, and thus $h^+(\Pi^C_{ce}) = h^{C+}_{nc}(\Pi)$. So, as desired, we obtain direct characterizations of both $h^+(\Pi^C)$ and $h^+(\Pi^C_{ce})$.

With Theorem 1 and known results about $h^+(\Pi^C)$ (Keyder, Hoffmann, and Haslum 2014), delete relaxation over explicit conjunctions behaves exactly as expected: $h^{C+} \geq h^C$, $h^{C+} \geq h^+$, $h^{C+} = \infty$ iff $h^C = \infty$, and $h^{C+} = h^*$ for appropriately chosen C.

$h^{\mathbf{FF}}(\Pi^C)$ w/o Compilation

We introduce a direct characterization, denoted $h^{C\text{FF}}$, of $h^{\text{FF}}(\Pi^C)$. We will build on the following characterization of standard relaxed plans: $\pi^{\text{FF}} := \pi(G)$ where $\pi(.)$ is a partial function on fact sets g that satisfies $\pi(g) =$

$$\begin{cases} \emptyset & g \subseteq I \\ \pi(g_r) \cup \{a\} \text{ where } a \in A, add(a) \cap g \neq \emptyset, & \\ \quad \text{and } h^1(g_r) < h^1(g \cap add(a)) & \text{else} \end{cases} \quad (3)$$

with $g_r := (g \setminus add(a)) \cup pre(a)$. As $g \cap add(a)$ always contains a fact p with $h^1(\{p\}) = h^1(g \cap add(a))$, this corresponds to relaxed plan extraction from the h^1 best-supporter function (Keyder and Geffner 2008). Note that $\pi(g)$ is undefined for *unsolvable subgoals* g where a *feasible* action a as requested does not exist. We say that $\pi^{\text{FF}} = \pi(G)$ is *supported* if the equation has a solution π whose domain contains G; similar for the equations below.

From Equation 3, we obtain $h^{C\text{FF}}$ by similar generalizations as we made to get from Equation 2 to h^{C+}:

Definition 3 *The* explicit-C FF heuristic $h^{C\text{FF}}$ *is defined as* $h^{C\text{FF}} := \infty$ *if* $h^C = \infty$, *else* $h^{C\text{FF}} := |\pi^{C\text{FF}}|$ *with* $\pi^{C\text{FF}} := \pi(\{G\})$ *where* $\pi(.)$ *is a partial function on conjunction sets* $\mathcal{G}$ *that satisfies* $\pi(\mathcal{G}) =$

$$\begin{cases} \emptyset & \forall g \in \mathcal{G} : g \subseteq I \\ \pi(\mathcal{G}_r) \cup \{(a, \mathcal{G}')\} \textit{ where } a \in A, & \\ \quad \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}, & \\ \quad \textit{and } h^C(\mathcal{G}_r) < h^C(\mathcal{G}') & \forall g \in \mathcal{G} : g \in C \\ \pi(\bigcup_{g \in \mathcal{G}} \{g' \subseteq g \mid g' \in C\}) & \textit{else} \end{cases}$$

Here, $\mathcal{G}_r := (\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g,a)\}$ *and* h^C *is extended to sets* $\mathcal{G}$ *of conjunctions by* $h^C(\mathcal{G}) := \max_{g \in \mathcal{G}} h^C$.

$h^{C\text{FF}}_{nc}$ *and* $\pi^{C\text{FF}}_{nc}$ *are defined in the same way except that* $\mathcal{G}_r := (\mathcal{G} \setminus \mathcal{G}') \cup \{R(g,a) \mid g \in \mathcal{G}'\}$.

The step from Equation 3 to Definition 3 should be clear given the discussion in the previous section. As before, atomic subgoals are conjunctions instead of single facts, each atomic subgoal must be supported correctly, and we minimize over choices of $\mathcal{G}'$. The equivalent of h^1 given conjunctions C is h^C. Maintaining a set of pairs $(a, \mathcal{G}')$ instead of a set of actions is required because the same action may be used several times, for different purposes (exactly as for the "action representatives" $a^{C'}$ in Π^C (Haslum 2012)).

Theorem 2 $\pi^{C\text{FF}}$ *is supported iff* $h^C < \infty$.

Proof Sketch: If $h^C = \infty$, then the top level subgoal already is unsolvable. Vice versa, if $h^C < \infty$, then the $\pi^{C\text{FF}}$ equation has a solution for $\pi(\{G\})$ even when restricting the choice of $\mathcal{G}'$, in the middle case, to singletons (i. e., to single conjunctions $\mathcal{G}' = \{g\}$). Intuitively, this is because h^C corresponds to reasoning over singleton conjunctions. ∎

Theorem 3 $\pi^{C\text{FF}}$*, if supported, is a relaxed plan for* Π^C.

Proof Sketch: Consider the sequence of pairs $(a_i, \mathcal{G}'_i)$ selected in a solution for $\pi^{C\text{FF}}$. It is easy to prove by induction over i that the sequence $a_i^{\mathcal{G}'_i}$ is a relaxed plan in Π^C. ∎

Similar proofs show that the same properties hold for $\pi^{C\text{FF}}_{nc}$.

There is an exponential number of choices for $\mathcal{G}'$ in the $\pi^{C\text{FF}}$ and $\pi^{C\text{FF}}_{nc}$ equations. The Π^C compilation can be understood as enumerating these choices explicitly in memory, via the action representatives $a^{C'}$. So how do we solve these equations in polynomial time? The answer is, for $\pi^{C\text{FF}}_{nc}$ the equation simplifies to a unique choice of $\mathcal{G}'$, and for $\pi^{C\text{FF}}$ we can approximate that choice greedily.

For $\pi^{C\text{FF}}_{nc}$, $\mathcal{G}'$ is *decomposable* in the sense that, for $\mathcal{G}' = \mathcal{G}'^1 \cup \mathcal{G}'^2$ and corresponding new generated subgoals $\mathcal{G}_r, \mathcal{G}^1_r, \mathcal{G}^2_r$, we have $\mathcal{G}_r = \mathcal{G}^1_r \cup \mathcal{G}^2_r$. So $\mathcal{G}'$ is *feasible*, i. e. $h^C(\mathcal{G}_r) < h^C(\mathcal{G}')$, iff each of its elements is, and we can restrict the choice of $\mathcal{G}'$ to be maximal, $\mathcal{G}' := \{g \in \mathcal{G} \mid R(g,a) \neq \bot, h^C(R(g,a)) < h^C(g)\}$. This essentially corresponds to what Π^C_{ce} achieves via conditional effects.

For Π^C, due to cross-context conditions, $\mathcal{G}'$ is not decomposable. However, to get a practical heuristic function, all we need is for the choice of $\mathcal{G}'$ to be complete (we do find feasible $\mathcal{G}'$ if there exists one) and sound (any $\mathcal{G}'$ we choose is feasible). Completeness is preserved even for singleton $\mathcal{G}'$, cf. Theorem 2. Soundness can be ensured easily during relaxed plan extraction. Our implementation keeps greedily adding new $g \in \mathcal{G}$ into $\mathcal{G}'$, until adding any single more g would result in $h^C(\{\bigcup_{g \in \mathcal{G}'} R(g,a)\}) \not< h^C(\mathcal{G}')$. It is interesting to note that an optimal selection of $\mathcal{G}'$ would be hard:

Theorem 4 *Given an integer* K*, in* $\pi^{C\text{FF}}$ *it is* **NP***-complete to decide whether there exists a feasible* $\mathcal{G}'$ *with* $|\mathcal{G}'| \geq K$.

Proof Sketch: Membership by guess and check. Hardness via a reduction of Hitting Set: Given a collection of sets $b \subseteq E$, the construction is such that choosing $\mathcal{G}'$ amounts to choosing $E' \subseteq E$, where E' is feasible iff there is no b with $b \subseteq E'$. $E' \setminus E$ then is a hitting set, and maximizing $|E'|$ is equivalent to finding a minimum-size such set. ■

This result somewhat "explains" the complexity difference between Π^C and Π^C_{ce}: Π^C_{ce} exploits decomposability for easy choice of $\mathcal{G}'$, whereas that choice incurs a hard problem in Π^C. The Π^C compilation inherently solves that problem enumeratively, which our $\pi^{C\text{FF}}$ algorithm avoids using a simple greedy algorithm. In other words: *There is no need to pre-generate all possible conjunction subsets an action could achieve. We can instead greedily tackle the subgoals that actually arise during relaxed plan extraction.*

Alcazar et al. (2013) earlier on introduced a heuristic "FF^m" which, like ours, handles explicit conjunctions without compilation. FF^m is like h^m in that it deals with all size-m conjunctions. Extending FF^m to arbitrary conjunction sets C, it corresponds in our notation to this equation:

$$\begin{cases} \emptyset & g \subseteq I \\ \pi(R(g,a)) \cup \{a\} & \\ \quad \text{where } a \in A, R(g,a) \neq \bot, \text{ and} & \\ \quad h^C(R(g,a)) < h^C(g \cap add(a)) & g \in C \\ \bigcup_{g' \subseteq g, g' \in C} \pi(g') & \text{else} \end{cases} \tag{4}$$

This suffers from two major weaknesses, with respect to our definition of $\pi^{C\text{FF}}$ above. (1) It uses "$\cup\{a\}$" instead of "$\cup\{(a,g)\}$", which bereaves the heuristic of almost all its power. It is now bounded from above by $|A|$, in contrast to h^{C+} which converges to h^*. (2) It restricts the choice of $\mathcal{G}'$ to singletons, and thus over-simplifies the subgoal-choice problem, running the risk of excessively long relaxed plans (or rather, the heuristic would be running that risk were it not trivialized as per (1)). Alcazar et al. effectively tackle Π^C_{nc} rather than Π^C because, with singleton $\mathcal{G}'$, cross-context conditions never occur. Indeed, using "$\cup\{(a,g)\}$" rather than '$\cup\{a\}$", Equation 4 is exactly what both $\pi^{C\text{FF}}$ and $\pi^{C\text{FF}}_{nc}$ simplify to when restricting the choice of $\mathcal{G}'$ to singletons.

Experiments

We implemented h^C, $h^{C\text{FF}}$, and $h^{C\text{FF}}_{nc}$ in FD (Helmert 2006). For h^C, we extended FF's counter-based implementation of relaxed planning graphs (Hoffmann and Nebel 2001). Instead of a counter for each action precondition, we maintain a counter for each pair (a, c) of action a and conjunction $c \in C$ where $R(c, a) \neq \bot$ and $R(c, a)$ contains no mutex fact pair. The last condition prunes useless counters, and is similar to *mutex pruning* of useless actions/conditional effects in Π^C/Π^C_{ce} as discussed by Keyder et al. (2014).

For each heuristic in our experiment, we implemented and ran three tie-breaking methods for relaxed plan extraction, FF's "difficulty" measure vs. arbitrary vs. random.[1] We ran all heuristics in FD's lazy-greedy search with a dual open queue for preferred operators, with time/memory bounds of 30 minutes/2 GB on Intel E5-2660 machines running at 2.20 GHz, on the IPC'11 and IPC'14 satisficing benchmarks.

[1] The performance variance over different random seeds is consistently small. Depending on the heuristic and domain, the difference to non-random tie-breaking, however, can be large.

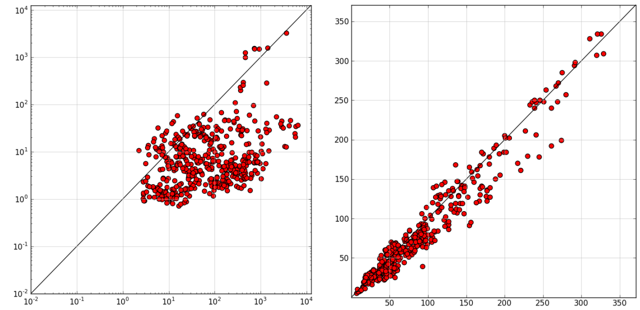

Figure 1: States per second (left) and initial state heuristic value (right) for $h^{C\text{FF}}$ (x-axis) vs. $h^{\text{FF}}(\Pi^C)$ (y-axis) on the IPC'11 and IPC'14 satisficing benchmarks, with large C ($x = \infty$). Both heuristics are run with FF's "difficulty" tie breaking. On the right-hand side, we omit data for the VisitAll and OpenStacks domains, as the heuristic values there are so large as to render the rest of the figure unreadable, and are basically identical anyhow.

The primary practical advantage of our work lies in the different computation of relaxed plans for Π^C, via greedy selection of $\mathcal{G}'$ in $h^{C\text{FF}}$ vs. enumerative such selection in $h^{\text{FF}}(\Pi^C)$. This corresponds to a different trade-off of heuristic function speed vs. accuracy. Consider Figure 1.

Keyder et al. generate C by iterative refinement of a relaxed plan for the initial state, stopping when either a bound x on the growth of the action set in Π^C relative to the original action set A, or a time-out of 15 minutes, is reached. To examine the effect of large C on $h^{C\text{FF}}$ vs. $h^{\text{FF}}(\Pi^C)$, we ran this process with $x = \infty$ (but keeping the time bound). As reported by Keyder et al., on IPC benchmarks, mutex pruning avoids the exponential blow-up of Π^C in $|C|$ completely. Still, an overhead remains, leading to the clear speed-up in Figure 1 (left). In Figure 1 (right), we see that the accuracy price paid, measured in terms of relaxed plan length on the initial state, is benign in comparison.

For $x = \infty$, this results in a substantial performance advantage, overall coverage being 256 with $h^{C\text{FF}}$ vs. 217 with $h^{\text{FF}}(\Pi^C)$. However, in most domains the best performance is obtained with small size bounds x. For $x = 2$, the overall best setting in Keyder et al.'s experiments, the comparison between $h^{C\text{FF}}$ and $h^{\text{FF}}(\Pi^C)$ is dominated by the variance over tie-breaking. The best overall coverage is 314 for $h^{C\text{FF}}$ (random tie-breaking) vs. 301 for $h^{\text{FF}}(\Pi^C)$ (difficulty tie-breaking). There are some cases in which increasing x helps, namely ChildSnack, CityCar, Maintenance, ParcPrinter, and Tetris; there are rare cases where this leads to improved overall best-possible performance with $h^{C\text{FF}}$ compared to $h^{\text{FF}}(\Pi^C)$. In Maintenance, for example, $h^{C\text{FF}}$ has coverage 12 for $x = 2$, 14 for $x = 4$, and 15 for $x = 8$, while the peak coverage obtained with $h^{\text{FF}}(\Pi^C)$ is 13 for $x = 32$.

For the comparison to $h^{\text{FF}}(\Pi^C_{ce})$, which like $h^{C\text{FF}}$ avoids the worst-case exponential blow-up of Π^C, consider Figure 2. The more complicated relaxed plan extraction in $h^{C\text{FF}}$ does result in a runtime overhead, reducing, for $x = \infty$, states per second typically by factors between 1 and 5. This pays off only if accounting for cross-context conditions gives an advantage in informativity, which in the IPC benchmarks happens rarely. In Maintenance, the peak coverage

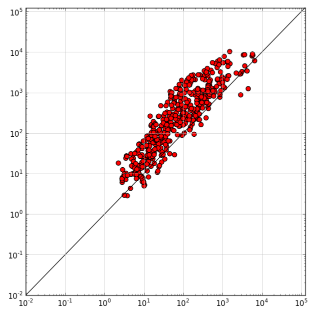

Figure 2: States per second for $h^{C\mathrm{FF}}$ (x-axis) vs. $h^{\mathrm{FF}}(\Pi^C_{ce})$ (y-axis) on the IPC'11 and IPC'14 satisficing benchmarks, with large C ($x = \infty$). Both heuristics are run with FF's "difficulty" tie breaking.

obtained with $h^{\mathrm{FF}}(\Pi^C_{ce})$ is 11 (compared to 15 with $h^{C\mathrm{FF}}$). For $x = 2$, as in the comparison $h^{C\mathrm{FF}}$ vs. $h^{\mathrm{FF}}(\Pi^C)$, the comparison $h^{C\mathrm{FF}}$ vs. $h^{\mathrm{FF}}(\Pi^C_{ce})$ is dominated by the tie-breaking differences. The same is true of $h^{C\mathrm{FF}}_{nc}$ vs. $h^{\mathrm{FF}}(\Pi^C_{ce})$.

Conclusion

Our direct formulation of delete relaxation over explicit conjunctions is nice in being less opaque than compilations, capturing such partial delete relaxation in terms of separate regression steps over conjunctive subgoals; and in enabling polynomial-time relaxed plans for Π^C. On IPC benchmarks, the benefits are visible but minor. It remains to be seen whether our formulation is fruitful for advanced research, such as deeper theoretical analyses of the approach.

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Proofs

Lemma 1 *For $C = \{\{p\} \mid p \in F\}$, $h^+ = h^{C+}$.*

Proof: With $C = \{\{p\} \mid p \in F\}$, the bottom case of the equation in Definition 2 splits $\mathcal{G}$ into single facts. Hence the internal structure of $\mathcal{G}$ – the fact subsets it contains – does not matter; it matters only which facts are contained in these fact subsets. We can thus perceive $\mathcal{G}$ as a set g of facts, re-writing the h^{C+} equation:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' \subseteq \{p \in g \mid R(\{p\},a) \neq \bot\}} & \\ \quad h((g \setminus g') \cup \bigcup_{p \in g'} R(\{p\}, a)) & \text{else} \end{cases}$$

For singleton subgoals $\{p\}$, regression trivializes, in the sense that $R(\{p\}, a)$ is defined iff $p \in add(a)$, and then $R(\{p\}, a) = pre(a)$. Therefore, choices of g' with $g' \subset \{p \in g \mid R(\{p\}, a) \neq \bot\}$, i. e., using a to achieve a strict subset of its possible benefit $g \cap add(a)$, can only lead to a larger subgoal $(g \setminus g') \cup pre(a)$. We can hence exclude such choices without affecting the minimum, and our equation simplifies to Equation 2, i. e.:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' = \{p \in g \mid R(\{p\},a) \neq \bot\}} & \\ \quad h((g \setminus g') \cup \bigcup_{p \in g'} R(\{p\}, a)) & \text{else} \end{cases}$$

Simplifying the notations, that equation becomes:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' = g \cap add(a)} & \\ \quad h((g \setminus g') \cup pre(a)) & \text{else} \end{cases}$$

As, for $g' = g \cap add(a)$, $g \setminus g' = g \setminus add(a)$, this is equivalent to:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' = g \cap add(a)} & \\ \quad h((g \setminus add(a)) \cup pre(a)) & \text{else} \end{cases}$$

This last equation clearly corresponds to h^+, proving the claim. ■

Theorem 1 $h^+(\Pi^C) = h^{C+}(\Pi)$.

Proof: Denoting by h^{1+} the application of h^{C+} to singleton subgoals only (which equals h^+ by Lemma 1), the claim is equivalent to $h^{1+}(\Pi^C) = h^{C+}(\Pi)$. We prove this property based on comparing two equations, characterizing $h^{1+}(\Pi^C)$ respectively $h^{C+}(\Pi)$.

For $h^{1+}(\Pi^C)$, by applying Definition 2 with singleton conjunctions to Π^C, we get (after easy simplifications):

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I^C \\ 1 + \min_{a^{C'} \in A^C, \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g, a^{C'}) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup pre(a^{C'})) & \forall g \in \mathcal{G} : |g| = 1 \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, |g'| = 1\}) & \text{else} \end{cases}$$

As argued in the proof of Lemma 1, with singleton subgoals there is no point in selecting non-maximal $\mathcal{G}'$, i. e., we need not choose less subgoals than $a^{C'}$ can support:

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I^C \\ 1 + \min_{a^{C'} \in A^C, \emptyset \neq \mathcal{G}' = \{g \in \mathcal{G} \mid R(g, a^{C'}) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup pre(a^{C'})) & \forall g \in \mathcal{G} : |g| = 1 \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, |g'| = 1\}) & \text{else} \end{cases}$$

Furthermore, there is no point in selecting C' larger than the set of subgoals that the original action a underlying $a^{C'}$ can support. Notating C' as $\mathcal{G}'$ to ease the notation below, we thus obtain our final equation, called *Equation I*, as:

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I^C \\ 1 + \min_{a^{\mathcal{G}'} \in A^C, \emptyset \neq \mathcal{G}' \subseteq \{\{\pi_g\} \in \mathcal{G} \mid R(g,a) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup pre(a^{\mathcal{G}'})) & \forall g \in \mathcal{G} : |g| = 1 \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, |g'| = 1\}) & \text{else} \end{cases}$$

For $h^{C+}(\Pi)$, we need to get rid of an irrelevant conceptual difference between Π^C and the equation defining h^{C+}: Whereas the latter splits up subgoals only if they are not contained in C, Π^C always includes all possible π-fluents, even from subgoals that are themselves already contained in C. Our new equation (called *Equation II*) modifies Definition 2 to do the same. We call $\mathcal{G}$ *completed* if, for all $g \in \mathcal{G}$, every $g' \subseteq g$ with $g' \in C$ is contained in $\mathcal{G}$ as well:

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g, a)\}) & \\ \qquad \mathcal{G} \text{ is completed and } \forall g \in \mathcal{G} : g \in C & \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, g' \in C\}) & \text{else} \end{cases}$$

This is equivalent because we only add subsumed subgoals.

Viewing each of Equations I and II as a tree whose root node is the "initializing call" containing the goal of the planning task, we show that the two trees are isomorphic. Namely, using the suffixes [I] and [II] to identify the tree, whenever the middle case applies we have:

$$(*) \ \mathcal{G}[I] = \{\{\pi_g\} \mid g \in \mathcal{G}[II]\}$$

This follows directly by inserting the respective definitions, which is simple but notationally cumbersome.

Consider the first time the middle case applies after the root call. For Equation I, the tree node prior to that application results from splitting up $\{G^C\}$ into singleton sets, one for every π-fluent. For Equation II, the node contains all C-subsets of $\{G\}$. Clearly, we have the desired correspondence (*).

For the induction step, assume that $\mathcal{G}[I]$ and $\mathcal{G}[II]$ are tree nodes to which the middle case applies, and with (*).

The choice of atomic subgoals g for $\mathcal{G}'$ is the same on both sides. By (*), for every atomic subgoal $g[I]$ in I, $g[I] = \{\pi_{g[II]}\}$ for $g[II] \in \mathcal{G}[II]$, and vice versa. And then, both equations apply the same criterion, namely $R(g[II], a) \neq \bot$ i. e. $add(a) \cap g[II] \neq \emptyset$ and $del(a) \cap g[II] = \emptyset$.

Say now that we select the same subgoal subset on both sides, i. e., $\mathcal{G}'[I] = \{\{\pi_{g[II]}\} \mid g[II] \in \mathcal{G}'[II]\}$, using action a in Equation II and action $a^{\mathcal{G}'[II]}$ in Equation I. Consider the regressed sets of subgoals, $\mathcal{G}_r[I]$ and $\mathcal{G}_r[II]$.

By Equation I and Definition 1, $\mathcal{G}'[I]$ is replaced in $\mathcal{G}_r[I]$ by (a) $(pre(a) \cup \bigcup_{g[II] \in \mathcal{G}'[II]} (g[II] \setminus add(a)))^C$. After an application of the bottom case (if needed), (a) turns into (a') $\{\{\pi_c\} \mid c \in C, c \subseteq pre(a) \cup \bigcup_{g[II] \in \mathcal{G}'[II]} (g[II] \setminus add(a))\}$.

On the other hand, by Equation II, $\mathcal{G}'[II]$ is replaced in $\mathcal{G}_r[II]$ by (b) $\bigcup_{g[II] \in \mathcal{G}'[II]} (pre(a) \cup (g[II] \setminus add(a)))$. After an application of the bottom case (if needed), (b) turns into (b') $\{c \in C, c \subseteq \bigcup_{g[II] \in \mathcal{G}'[II]}(pre(a) \cup (g[II] \setminus add(a)))\}$. Because $\mathcal{G}'[II] \neq \emptyset$, (a') and (b') are, again, in correspondence (*), which concludes our argument. ∎

Lemma 2 *For $C = \{\{p\} \mid p \in F\}$, $h^+ = h^{C+}_{nc}$.*

Proof: The proof is the same as that of Lemma 1, except that the first equation now is:

$$\begin{cases} 0 & g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq g' \subseteq \{p \in g \mid R(\{p\},a) \neq \bot\}} & \\ \quad h((g \setminus g') \cup \{R(g,a) \mid g \in \mathcal{G}'\}) & \text{else} \end{cases}$$

Again, because regression trivializes on singleton subgoals $\{p\}$ there is no point in choosing $g' \subset \{p \in g \mid R(\{p\}, a) \neq \bot\}$, i. e., using a to achieve a strict subset of its possible benefit $g \cap add(a)$, because that can only lead to a larger subgoal $(g \setminus g') \cup pre(a)$. Furthermore, as $R(g, a) = pre(a)$ for all $g \in \mathcal{G}$, there is no difference between $\{R(g, a) \mid g \in \mathcal{G}'\}$ and $\bigcup_{p \in g'} R(\{p\}, a)$, so the equation simplifies to Equation 2. After that, the proof is the same as before. ∎

Theorem 5 $h^+(\Pi^C_{nc}) = h^{C+}_{nc}(\Pi)$.

Proof: As in the proof of Theorem 1, denoting by h^{1+} the application of h^{C+}_{nc} to singleton subgoals only (which equals h^+ by Lemma 2), the claim is equivalent to $h^{1+}(\Pi^C) = h^{C+}(\Pi)$. We prove this property based on comparing two equations, characterizing $h^{1+}(\Pi^C_{nc})$ respectively $h^{C+}_{nc}(\Pi)$.

Equation II is as before, except for the difference in precondition subgoal generation, using $\{R(g, a) \mid g \in \mathcal{G}'\}$ in the middle case instead of $\{\bigcup_{g \in \mathcal{G}'} R(g, a)\}$:

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I \\ 1 + \min_{a \in A, \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup \{R(g,a) \mid g \in \mathcal{G}'\}) & \\ \qquad \mathcal{G} \text{ is completed and } \forall g \in \mathcal{G} : g \in C & \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, g' \in C\}) & \text{else} \end{cases}$$

Equation I is exactly as before (the difference in precondition subgoal generation does not show as we're dealing with singleton conjunctions only):

$$\begin{cases} 0 & \forall g \in \mathcal{G} : g \subseteq I^C \\ 1 + \min_{a^{\mathcal{G}'} \in A^C, \emptyset \neq \mathcal{G}' \subseteq \{\{\pi_g\} \in \mathcal{G} \mid R(g,a) \neq \bot\}} & \\ \quad h((\mathcal{G} \setminus \mathcal{G}') \cup pre(a^{\mathcal{G}'})) & \forall g \in \mathcal{G} : |g| = 1 \\ h(\bigcup_{g \in \mathcal{G}} \{g' \mid g' \subseteq g, |g'| = 1\}) & \text{else} \end{cases}$$

Again, we view each of Equations I and II as a tree, and show that:

$$(*)\ \mathcal{G}[I] = \{\{\pi_g\} \mid g \in \mathcal{G}[II]\}$$

For the first time the middle case applies after the root call, nothing changes with respect to the proof of Theorem 1. Also, assuming for the induction step that $\mathcal{G}[I]$ and $\mathcal{G}[II]$ are tree nodes to which the middle case applies, and with (*), the supportable atomic subgoals g for $\mathcal{G}'$ are the same on both sides.

Say now that we select the same subgoal subset on both sides, i. e., $\mathcal{G}'[I] = \{\{\pi_{g[II]}\} \mid g[II] \in \mathcal{G}'[II]\}$, using action a in Equation II and action $a^{\mathcal{G}'[II]}$ in Equation I. Consider the regressed sets of subgoals, $\mathcal{G}_r[I]$ and $\mathcal{G}_r[II]$.

By Equation I and Definition 1, $\mathcal{G}'[I]$ is replaced in $\mathcal{G}_r[I]$ by (a) $pre(a)^C \cup \bigcup_{g[II] \in \mathcal{G}'[II]}(pre(a) \cup (g[II] \setminus add(a)))^C$. After an application of the bottom case (if needed), (a) turns into (a') $\{\{\pi_c\} \mid c \in C, c \subseteq pre(a)$ or ex. $g[II] \in \mathcal{G}'[II]$ s.t. $c \subseteq pre(a) \cup (g[II] \setminus add(a))\}$.

On the other hand, by Equation II, $\mathcal{G}'[II]$ is replaced in $\mathcal{G}_r[II]$ by (b) $\{pre(a) \cup (g[II] \setminus add(a)) \mid g[II] \in \mathcal{G}'[II]\}$. After an application of the bottom case (if needed), (b) turns into (b') $\{c \in C,$ ex. $g[II] \in \mathcal{G}'[II]$ s.t. $c \subseteq pre(a) \cup (g[II] \setminus add(a))\}$. Because $\mathcal{G}'[II] \neq \emptyset$, (a') and (b') are, again, in correspondence (*), which concludes our argument. ∎

Theorem 2 *$\pi^{C\text{FF}}$ is supported iff $h^C < \infty$.*

Proof: The direction from left to right is easy: If $h^C = \infty$, then there exists $g' \subseteq g, g' \in C$ so that $h^C(g') = \infty$. Therefore, in the equation defining $\pi^{C\text{FF}}$, after the bottom case was applied to $\mathcal{G} = \{G\}$, $h^C(\mathcal{G}_r) = \infty$ for all choices of $a, \mathcal{G}'$, and hence $\{G\}$ cannot be in the domain of any solution π.

For the direction from right to left, we consider a simpler version of the equation defining $\pi^{C\text{FF}}$, restricting the choice of $\mathcal{G}'$ to singletons, i. e.:

$$\begin{cases} \emptyset & \forall g \in \mathcal{G} : g \subseteq I \\ \pi(\mathcal{G}_r) \cup \{(a, \mathcal{G}')\} \text{ where } a \in A, & \\ \quad \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}, |\mathcal{G}'| = 1, & \\ \quad \text{and } h^C(\mathcal{G}_r) < h^C(\mathcal{G}') & \forall g \in \mathcal{G} : g \in C \\ \pi(\bigcup_{g \in \mathcal{G}} \{g' \subseteq g \mid g' \in C\}) & \text{else} \end{cases}$$

with $\mathcal{G}_r := (\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g, a)\}$. As we always select a singleton $\mathcal{G}'$ here, we can modify the recursion to proceed over singleton sets of atomic subgoals:

$$\begin{cases} \emptyset & \forall g \in \mathcal{G} : g \subseteq I \\ \pi(\mathcal{G}_r) \cup \{(a, \mathcal{G}')\} \text{ where } a \in A, & \\ \quad \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}, |\mathcal{G}'| = 1, & \\ \quad \text{and } h^C(\mathcal{G}_r) < h^C(\mathcal{G}') & \mathcal{G} = \{g\}, g \in C \\ \pi(\bigcup_{g \in \mathcal{G}} \{g' \subseteq g \mid g' \in C\}) & \text{else} \end{cases}$$

with $\mathcal{G}_r := (\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g, a)\}$. Simplifying the notation, given that our subgoals are now just fact conjunctions, we get exactly Equation 4:

$$\begin{cases} \emptyset & g \subseteq I \\ \pi(R(g,a)) \cup \{(a, g)\} & \\ \quad \text{where } a \in A, R(g,a) \neq \bot, \text{ and} & \\ \quad h^C(R(g,a)) < h^C(g \cap add(a)) & g \in C \\ \bigcup_{g' \subseteq g, g' \in C} \pi(g') & \text{else} \end{cases}$$

Comparing this to Equation 1, it is obvious that $\pi(G)$ is supported iff $h^C < \infty$.

Getting back the theorem's claim, from right to left, if $h^C < \infty$ then $\pi(G)$ in Equation 4 is supported, and because Equation 4 is a restricted version of the equation defining $h^{C\text{FF}}$, in that latter equation $\pi(G)$ is supported as well, so $\pi^{C\text{FF}}$ is supported as required. ∎

Theorem 6 *$\pi^{C\text{FF}}_{nc}$ is supported iff $h^C < \infty$.*

Proof: The direction from left to right is exactly as in the proof of Theorem 2. The direction from right to left follows because the equation defining h^{CFF}_{nc} simplifies to the exact same equation, i. e. to Equation 4, when restricting the choice of $\mathcal{G}'$ to singletons. ■

Theorem 3 *$\pi^{C\text{FF}}$, if supported, is a relaxed plan for Π^C.*

Proof: Recall the definition of $\pi^{C\text{FF}}$: $\pi^{C\text{FF}} := \pi(\{G\})$ where $\pi(.)$ is a partial function on conjunction sets $\mathcal{G}$ that satisfies $\pi(\mathcal{G}) =$

$$\begin{cases} \emptyset & \forall g \in \mathcal{G} : g \subseteq I \\ \pi(\mathcal{G}_r) \cup \{(a, \mathcal{G}')\} \text{ where } a \in A, & \\ \quad \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}, & \\ \quad \text{and } h^C(\mathcal{G}_r) < h^C(\mathcal{G}') & \forall g \in \mathcal{G} : g \in C \\ \pi(\bigcup_{g \in \mathcal{G}} \{g' \subseteq g \mid g' \in C\}) & \text{else} \end{cases}$$

with $\mathcal{G}_r := (\mathcal{G} \setminus \mathcal{G}') \cup \{\bigcup_{g \in \mathcal{G}'} R(g,a)\}$.

Assume a solution for $\pi(\{G\})$. Denote by $\langle (a_0, \mathcal{G}'_0), \dots, (a_{n-1}, \mathcal{G}'_{n-1}) \rangle$ the inverted sequence of action/subgoal-set pairs selected in the recursive steps leading to π, i. e., deeper recursion steps correspond to smaller indices, $(a_0, \mathcal{G}'_0)$ is the pair whose selection lead to the terminating top case, and $(a_{n-1}, \mathcal{G}'_{n-1})$ is the pair selected at the first occurence of the middle case. We show that, when selecting $C'_i := \mathcal{G}'_i$, $\langle a_0^{C'_0}, \dots, a_{n-1}^{C'_{n-1}} \rangle$ is a (relaxed) plan for Π^C. This shows the claim as π contains all elements of $\langle (a_0, \mathcal{G}'_0), \dots, (a_{n-1}, \mathcal{G}'_{n-1}) \rangle$ once, and applying the same $a^{C'}$ twice obviously is redundant.

Denote by $\mathcal{G}_i$, for $1 \leq i \leq n$, the subgoal tackled by the selection of $(a_{i-1}, \mathcal{G}'_{i-1})$ in the middle case, and denote by $\mathcal{G}_0$ the final subgoal tackled by the top case. Denote by s_i the state resulting from applying $\langle a_0^{\mathcal{G}'_0}, \dots, a_{i-1}^{\mathcal{G}'_{i-1}} \rangle$ in Π^C. We show by induction over i that *(*)* $\{pi_g \mid g \in \mathcal{G}_i\} \subseteq s_i$. For $i = n$, this shows that $s_n \supseteq G^C$ as desired.

Induction base case, $i = 0$: Here, (*) follows directly from definition because, the top case having fired on $\mathcal{G}_0$, for all $g \in \mathcal{G}_0$ we have that $g \subseteq I$, and hence $\pi_g \in I^C$.

For the induction step, assume that (*) is true up to i. We show that it holds for $i+1$. By construction, $\mathcal{G}_i$ is generated by applying the bottom case (if needed) to the subgoal $\mathcal{G}_r = (\mathcal{G}_{i+1} \setminus \mathcal{G}'_i) \cup \{\bigcup_{g \in \mathcal{G}'_i} R(g, a_i)\}$. The left-hand side subset of $\mathcal{G}_r$ delivers, in the bottom case, the atomic subgoals $\bigcup_{g \in \mathcal{G}_{i+1} \setminus \mathcal{G}'_i} \{g' \mid g' \subseteq g, g' \in C\}$; denote that set of atomic subgoals by LH. The right-hand side of $\mathcal{G}_r$ delivers, in the bottom case, the atomic subgoals $\{g' \mid g' \subseteq \bigcup_{g \in \mathcal{G}'_i} R(g, a_i), g' \in C\}$; denote that set of atomic subgoals by RH. As $\mathcal{G}'_i \neq \emptyset$, and as $R(g, a_i) = (g \setminus add(a_i)) \cup pre(a_i)$ for each $g \in \mathcal{G}'_i$, we have $RH = \{g' \mid g' \subseteq pre(a) \cup \bigcup_{g \in \mathcal{G}'_i} (g \setminus add(a)), g' \in C\}$.

By construction, $\mathcal{G}_i = LH \cup RH$, and by induction hypothesis we have (*) $\{pi_{g'} \mid g' \in LH \cup RH\} \subseteq s_i$. Consider now $\mathcal{G}_{i+1}$ and s_{i+1}. First, those atomic subgoals not achieved by $(a_i, \mathcal{G}'_i)$, namely $\mathcal{G}_{i+1} \setminus \mathcal{G}'_i$, are tackled by LH: As $LH = \bigcup_{g \in \mathcal{G}_{i+1} \setminus \mathcal{G}'_i} \{g' \mid g' \subseteq g, g' \in C\}$, and because each atomic subgoal $g \in \mathcal{G}_{i+1}$ in the middle case must be an element of C itself, with (*) $\{pi_{g'} \mid g' \in LH\} \subseteq s_i$ we have that $\{pi_g \mid g \in \mathcal{G}_{i+1} \setminus \mathcal{G}'_i\} \subseteq s_i$. As the planning is delete-free this immediately yields $\{pi_g \mid g \in \mathcal{G}_{i+1} \setminus \mathcal{G}'_i\} \subseteq s_{i+1}$. Second, those atomic subgoals that *are* achieved by $(a_i, \mathcal{G}'_i)$, namely $\mathcal{G}'_i$, clearly will be true in s_{i+1} as well: This is simply because $add(a^{\mathcal{G}'_i}) = \{\pi_g \mid g \in \mathcal{G}'_i\}$.

It remains to show that $a^{\mathcal{G}'_i}$ is applicable in s_i. Its precondition is $(pre(a) \cup \bigcup_{g \in \mathcal{G}'_i} (g \setminus add(a)))^C$, that is, $pre(a^{\mathcal{G}'_i}) = \{\pi_{g'} \mid g' \subseteq pre(a) \cup \bigcup_{g \in \mathcal{G}'_i} (g \setminus add(a)), g' \in C\}$. This is exactly $pre(a^{\mathcal{G}'_i}) = \{\pi_{g'} \mid g' \in RH\}$, so we are done by (*) $\{pi_{g'} \mid g' \in RH\} \subseteq s_i$ which concludes the proof. ■

Theorem 7 *$\pi^{C\text{FF}}_{nc}$, if supported, is a relaxed plan for Π^C_{nc}.*

Proof: Recall the definition of $\pi^{C\text{FF}}_{nc}$: $\pi^{C\text{FF}}_{nc} := \pi(\{G\})$ where $\pi(.)$ is a partial function on conjunction sets $\mathcal{G}$ that satisfies $\pi(\mathcal{G}) =$

$$\begin{cases} \emptyset & \forall g \in \mathcal{G} : g \subseteq I \\ \pi(\mathcal{G}_r) \cup \{(a, \mathcal{G}')\} \text{ where } a \in A, & \\ \quad \emptyset \neq \mathcal{G}' \subseteq \{g \in \mathcal{G} \mid R(g,a) \neq \bot\}, & \\ \quad \text{and } h^C(\mathcal{G}_r) < h^C(\mathcal{G}') & \forall g \in \mathcal{G} : g \in C \\ \pi(\bigcup_{g \in \mathcal{G}} \{g' \subseteq g \mid g' \in C\}) & \text{else} \end{cases}$$

with $\mathcal{G}_r := (\mathcal{G} \setminus \mathcal{G}') \cup \{R(g,a) \mid g \in \mathcal{G}'\}$.

The proof of Theorem 3 remains valid exactly as written, except that now $RH = \{g' \mid \text{ex. } g \in \mathcal{G}' \text{ s.t. } g' \subseteq R(g, a_i), g' \in C\}$. Because $\mathcal{G}'_i \neq \emptyset$, and as $R(g, a_i) = (g \setminus add(a_i)) \cup pre(a_i)$ for each $g \in \mathcal{G}'_i$, we have $RH = \{g' \mid g' \subseteq pre(a) \text{ or ex. } g \in \mathcal{G}' \text{ s.t. } g' \subseteq pre(a) \cup (g \setminus add(a)), g' \in C\}$. The precondition of $a^{\mathcal{G}'_i}$ is $pre(a)^C \cup \bigcup_{g \in \mathcal{G}'_i} (pre(a) \cup (g \setminus add(a)))^C$, that is, $pre(a^{\mathcal{G}'_i}) = \{\pi_{g'} \mid g' \subseteq pre(a) \text{ or ex. } g \in \mathcal{G}' \text{ s.t. } g' \subseteq pre(a) \cup (g \setminus add(a)), g' \in C\}$. This is exactly $pre(a^{\mathcal{G}'_i}) = \{\pi_{g'} \mid g' \in RH\}$, so again we are done by $\{pi_{g'} \mid g' \in RH\} \subseteq s_i$. ■

Theorem 4 *Given an integer K, in $\pi^{C\text{FF}}$ it is **NP**-complete to decide whether there exists a feasible $\mathcal{G}'$ with $|\mathcal{G}'| \geq K$.*

Proof: Membership is obvious by guess and check. For hardness, we show a polynomial reduction from the Hitting Set problem with a set B of subsets $b \subseteq E$, the question being whether there exists a hitting set of size at most L. Say that $E = \{e_1, \dots, e_n\}$. We construct a planning task as follows. $F := E \cup \{p_0, p_1, p_2\} \cup \{g_1, \dots, g_n\}$, $I := \emptyset$, $G := \{g_1, \dots, g_n\}$. The action set contains an action a_{e_i} for every $e_i \in E$ with $add(a_{e_i}) = \{e_i\}$, $pre(a_{e_i}) = \emptyset$, and $del(a_{e_i}) = \{p_2\} \cup \{e_j \mid \text{ex. } b \in B : \{e_i, e_j\} \subseteq b\}$. Furthermore, the action set contains a^p_1 with precondition p_0, add p_1, and empty delete, as well as a^p_2 with precondition p_1, add p_2, and empty delete. Finally, the action set contains the actions $a^g_1, \dots, a^g_n$ where $pre(a^g_i) = \{e_i, p_2\}$, $add(a^g_i) = \{g_i\}$, and the delete is empty. We set $C := \{\{p\} \mid p \in F\} \cup B \cup \{\{e_i, p_2\} \mid e_i \in E\}$.

We think of h^C now in terms of a (C-)relaxed planning graph (RPG), where layer t corresponds to the conjunctions g with $h^C(g) \leq t$. None of the conjunctions $b \in B$ can be achieved, as there exists no action through which b can be regressed. However, all the facts $e_i \in E$ can be achieved in isolation. Consider layer 1 of the RPG. The key property we exploit below is that *(*) any subset* $E' = \{e_1, \dots, e_k\} \subseteq E$ *is feasible at layer* 1 *iff there does not exist* $b \in B$ *s.t.* $b \subseteq E'$. From right to left, if $b \subseteq E'$ then E' cannot be achieved simply because $h^C(b) = \infty$. Vice versa, say there does not exist $b \in B$ s.t. $b \subseteq E'$. Then $g' \subseteq E, g' \in C$ is just the set of singleton-conjunction π-fluents π_{e_i}, and we get $h^C(E) = 1$ as each π_{e_i} is achieved by a single action.

At RPG layer 1, we can apply a_2^p. As each e_i is already present, we get each of the conjunctions $\{e_1, p_2\}, \dots, \{e_n, p_2\}$ at layer 2. With this, the a_i^g actions become feasible, so that the goal is reached at layer 3.

Consider now relaxed plan extraction. To get the goal, we must select all a_i^g actions. Say all those are selected in sequence. Then we get the subgoal $\{\{e_1, p_2\}, \dots, \{e_n, p_2\}\}$ at layer 2 (plus the subsumed singleton conjunctions, which we omit for readability). The only action through which these can be regressed is a_2^p: recall that the a_{e_i} actions delete p_2. But what is the maximal subset $\mathcal{G}' := \{\{e_{i_1}, p_2\}, \dots, \{e_{i_k}, p_2\}\} \subseteq \{\{e_1, p_2\}, \dots, \{e_n, p_2\}\}$ that we can choose? Any such subset yields the new generated subgoal $\{e_{i_1}, \dots, e_{i_k}, p_1\}$ at RPG layer 1. Here p_1 is achieved by a_1^p which does not interact with anything so is not critical: $h^C(\{e_{i_1}, \dots, e_{i_k}, p_1\}) = h^C(\{e_{i_1}, \dots, e_{i_k}\})$. Denote $E' := \{e_{i_1}, \dots, e_{i_k}\}$. As per (*), E' is feasible at RPG layer 1 iff there does not exist $b \in B$ s.t. $b \subseteq E'$. But then, consider $E \setminus E'$. By construction, this is a hitting set iff E' is feasible: if $E \setminus E'$ is a hitting set then no b can be fully contained in E', and if no b is fully contained in E' then $E \setminus E'$ must hit every b. Setting $K := n - L$, we thus get that there exists a feasible $\mathcal{G}'$ with $|\mathcal{G}'| \geq K$ iff there exists a feasible E' with $|E'| \geq n - L$ iff there exists a hitting set of size $\leq n - (n - L) = L$. This concludes the proof. ■